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REMARKS ON PROFESSOR BOOLE'S MATHEMATICAL THEORY OF THE LAWS OF THOUGHT.

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In a recent issue we announced the death of Professor George Boole, of Queen's College, Cork, a man of varied and profound acquirements, and of singular originality of mind. The work on which his fame will mainly rest is undoubtedly his "Investigation of the Laws of Thought, on which are founded the Mathematical Theories of Logic and Probabilities." We have long purposed to call attention to this remarkable production, though various circumstances have hitherto prevented us from doing so. The present seems a suitable occasion for testifying our admiration of the genius of the deceased philosopher, and, at the same time, endeavouring to give a brief account, inadequate as it must necessarily be, of what may be termed his Mathematico-logical speculations.

The primary, though not the exclusive, design of the "Investigation," is to express in the symbolical language of a Calculus, the fundamental Laws of Thought, and upon this foundation to establish the science of Logic and construct its method.

The elementary symbols of Professor Boole's Calculus are of three kinds: 1st. Literal symbols, as x, y, &c., representing the objects of our conceptions; 2nd. Signs of operation, as $+$, $-$, $\times$; and 3rd, the sign of identity, =. The sign + is used to express the mental operation by which parts (of extensive quantity) are collected into a whole. For instance, if x represent *animals*, and y *vegetables*, $x + y$ will represent the class made up of *animals and vegetables together*. On the other hand, the sign — is used to express the mental operation of separating a whole (of extensive quantity) into its parts. Thus, x representing *human beings*, and y representing *negroes*, $x - y$ will represent *all human beings except negroes*. With regard to the sign $\times$, $x \times y$ or $x\,y$ (as it may be written) is used to denote those objects which belong at once to the class x and to the class y; just as, in common language, the expression *dark-waters* denotes those objects which are at once *dark* and *waters*. Hence we obtain a method of representing a concept taken particularly. For, if x denote *men*, then, since *some men* may be viewed as those who besides belonging to the class x belong also to some other class v, *some men* will be denoted by $v\,x$. In general,

$$v\,x = \text{some } x. \quad\text{.........................(1)}$$

It can easily be shown, that, as in Algebra, so in the logical system which we are describing, the literal symbols, x, y, &c., are commutative; that is,

$$x\,y = y\,x; \quad\text{..............................(2)}$$

and that they are also distributive; that is,

$$z\,(x \pm y) = z\,x \pm z\,y \quad\text{..................(3)}$$

Another relation between Algebra and the Logical System under consideration is, that, in the latter as well as in the former, a literal symbol may be transposed from one side of an equation to the other by changing the sign of operation, + or —. But there is an important relation which subsists in the science of Thought, and not generally in Algebra, namely,

$$x^2 = x \quad\text{...................................(4)}$$

That this is true in the Logical system, is plain; for x^2, which is another form of $x\,x$, denotes (by definition) those things which belong at once to the class x and to the class x; that is, it denotes simply those things which belong to the class x; and it is therefore identical with x. But though the equation (4) does not generally subsist in Algebra, it subsists when x is unity or zero. If, therefore, we take the science of Algebra with the limitation that its unknown quantities can receive no values distinct from unity and zero, the analogy between the two sciences will still be preserved.

It is necessary to observe that unity and zero (1 and 0) are virtually included by Professor Boole among his literal symbols. Of course we can give 1 and 0 any meaning we please, provided the meaning once imposed on them be rigidly adhered to. By 0, then, Professor Boole understands Nothing—a class (if the expression may be permitted) in which no object whatever is found. On the other hand, by 1 he understands the universe of conceivable objects. Thus 1 and 0 are at two opposite poles; the former including every thing in its extension; the latter, nothing. The meaning which has been affixed to 1 and 0 preserves, in the Logical system as in Algebra, the equations,

$$\left.\begin{array}{r} 1 \times x = x, \\ \text{and, } 0 \times x = 0; \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

for, the meaning of the former is, that objects which are common to the universe and to the class x are identical with those which constitute the class x; and the latter means, that there are no objects which are common to a class in which nothing is found and to a class x: both of which propositions are self-evident. From the meaning affixed to 1, we see what the meaning of $1 - x$ must be. In fact, x and $1 - x$ are logical contradictories, the latter denoting all conceivable objects except those which belong to the former; so that

$$1 - x = \text{not } x \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

This value of the symbol 1 being admitted, we can, by the principles of transposition and distribution [see (3)] reduce equation (4) to the form,

$$x(1 - x) = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$

The law here expressed, which is termed the Law of Duality, plays a most important part in the development of logical functions, and in the elimination of symbols. In fact, it may be described as the germ out of which Professor Boole's whole system is made to unfold itself.

Having shown how concepts, whether taken universally or particularly, are represented, and also how the contradictory of a concept is represented, we have next to notice the manner of expressing judgments. All judgments are regarded by our author as affirmative; the negation, in those which are commonly called negative, being attached by him to the predicate. But an affirmative judgment is nothing else than an assertion, through immediate comparison, of the identity of concepts. Suppose, therefore, that we are required to express the judgment, "Some stones are precious." Let x denote *stones;* and y, *precious*. The proposition means, that some stones are identical with some precious things. Consequently, its symbolical expression [see (1)] is,

$$vx = vy.$$

If the judgment to be represented had been, "Some stones are not precious," its expression would [see (6)] have been

$$vx = v(1 - y).$$

These examples in the meantime may suffice. More complicated forms will present themselves afterwards.

With the few simple preliminary explanations which have been given, and which were necessary to render intelligible some of the criticisms presently to be offered, we are now prepared to state the view which our author takes of the science of Logic. Logic he regards as the science of Inference; and the problem which it seeks to solve is this: Given certain relations among any number of concepts (x, y, z, &c.), it is required to find what inferences can be drawn regarding any one of these or regarding a given function of any one of them. A properly constructed science of Logic would require to solve this problem adequately, and by a definite and invariable method. Now, Professor Boole claims that the view which he presents of the problem which Logic has to solve, is both deeper and broader than that commonly taken; and he claims at the same time that he has devised an adequate method, different from all existing methods, for solving this problem, and that his method is one of definite and invariable application.

The objections brought against the logic of the schools, that it is neither sufficiently deep nor sufficiently broad, will probably take our readers by surprise. It is not difficult to understand how a question might be raised as to the practical utility of the scholastic logic; but most persons who have examined the subject will be ready to admit, both that the scholastic logic is well founded, and that, when properly developed from its first principles, it forms a complete and perfect system. In the opinion of our author, however, it is so defective in its foundation, and so incomplete in its superstructure, as not to be entitled to the name of a science. "To what final conclusions," he says, "are we then led respecting the nature and extent of the scholastic logic? I think to the following: that it is not a science, but a collection of scientific truths, too incomplete to form a system of themselves, and not sufficiently fundamental to serve as the foundation upon which a perfect system may rest."

In order that it may be understood in what sense it is held that *the foundation of the scholastic logic is defective,* we make two other quotations. "That which may be regarded as essential in the spirit and procedure of the Aristotelian, and of all cognate systems of logic, is the attempted classification of the allowable forms of inference, and the distinct reference of those forms, collectively or individually, to some general principle of an axiomatic nature, such as the Dictum of Aristotle." Again: "Aristotle's Dictum de omni et nullo is a self-evident principle, but it is not found among those ultimate laws of the reasoning faculty to which all other laws, however plain and self-evident, admit of being traced, and from which they may in strictest order of scientific evolution be deduced. For though of every science the fundamental truths are usually the most simple of apprehension, yet is not that simplicity the criterion by which their title to be regarded as fundamental must be judged. This must be sought for in the nature and extent of the structure which they are capable of supporting. Taking this view, Leibnitz appears to me to have judged correctly when he assigned to the principle of contradiction a fundamental place in logic; for we have seen the consequences of that law of thought of which it is the axiomatic expression." The sum of what is contained in these passages, in so far as they bear on the point before us, is, 1st, That the foundation of the Aristotelian, and of all cognate systems of logic, is some such canon as the Dictum; 2nd, That that canon, and other maxims of a like description, though self-evident, are not deep enough to serve as a basis for a science of logic in which all the forms of thought are to be exhibited; and, 3rd, That the only principle sufficiently fundamental to form the basis of a complete science of logic is the principle of contradiction. Now what is the real state of the case? Nothing is more certain than that the Dictum was not considered by Aristotle as either the exclusive or the ultimate foundation of his logical system. Not the exclusive foundation; for, as a matter of fact, many of the forms of thought embraced in the Aristotelian logic receive no direct warrant from the Dictum, but can be derived from it only by the aid of the principle of contradiction. Not the ultimate foundation; for what is the Dictum, but a particular case of a more comprehensive, and (in this sense) more fundamental, law? Aristotle saw this, and has expressed it as clearly as any man that ever lived. "It is manifest," he says, "that no one can conceive to himself that the same thing can at once be and not be, for thus he would hold repugnant opinions, and subvert the reality of truth. Wherefore, all who attempt to demonstrate, reduce everything to this as the ultimate doctrine; for this is by nature the principle of all other axioms."

Professor Boole's acceptance of the Leibnitzian maxim (though it was much older than Leibnitz) that the true foundation of the science of logic is the principle of contradiction, has the appearance of being at variance with some extraordinary statements which he elsewhere makes, to the effect that the principle of contradiction is a consequence of the law of duality. We may remind our readers that the law of duality [see (4) and (7)] is substantially the principle out of which all the details of Professor Boole's own doctrine are evolved. Now, under the influence of what was, perhaps, not an unnatural desire to vindicate for his system a peculiar depth of foundation, Professor Boole has been betrayed into observations by which his fame as a philosophic thinker must be seriously affected. For instance: "that axiom of metaphysicians which is termed the principle of contradiction, and which affirms that it is impossible for any being to possess a quality and at the same time not to possess it, is a consequence of the fundamental law of thought, whose expression is $x^2 = x$." And again: "the above interpretation has been introduced, not on account of its immediate value in the present system, but as an illustration of a significant fact in the philosophy of the intellectual powers, viz., that what has commonly been regarded as the fundamental axiom of metaphysics is but the consequence of a law of thought, mathematical in its form." In thus speaking of the principle of contradiction as a consequence of the law of duality, Professor Boole seems to take away the fundamental character of the principle of contradiction; for, if that principle be, in the proper sense of the term, a consequence of something else, it cannot be itself truly fundamental. Yet, as we have seen, Professor Boole admits that it is the real and deepest foundation of the science of logic. What, then, does he mean? On the one hand, he certainly does not intend to deny that the principle of contradiction is self-evident. On the other hand, it is plain that he does hold that the principle of contradiction can be deduced from the law of duality. But (we ask) how? Can the principle of contradiction be deduced from the law of duality, without our assuming the principle of contradiction itself as the basis of the deduction? This would be absurd; for a conclusion can be established in no other way than by pointing out that the supposition of its being false involves a contradiction. In the particular case before us, the equation $x(1-x)=0$, which is that expression of the law of duality in which the principle of contradiction is regarded as being brought to light, is only reached by a process of reasoning, every step of which takes the principle of contradiction for granted. The only interpretation, therefore, which Professor Boole's words can bear, unless we give them a meaning palpably absurd, is, that a formula, which we are enabled to state by assuming the law of contradiction, contains a symbolic representation of that law. This hardly seems to us a very significant fact in the philosophy of the intellectual powers. If indeed the formula in question could be shown to represent some law of thought of wider application than the law of contradiction, that would be a very significant fact. But such is not the case. The equation $x(1-x)=0$ is just the law of contradiction symbolically expressed: neither more nor less.

The Aristotelian logic is charged with being *incomplete*, as well as with being not sufficiently fundamental. By this our author does not mean that Aristotle and his followers have casually omitted some forms of thought which their system ought to have embraced: had they done so, the fault would have been chargeable—not upon the system, but upon its expounders; but he means, that, from the very nature of the system, there is an indefinite variety of problems belonging to the science of inference, which their system is incapable of solving, or for the solution of which at all events it furnishes no definite and certain method.

It will be observed that there are two questions here, which, as radically distinct from one another, require to be considered separately: the one being, whether the Aristotelian logic is capable of solving all the problems belonging to the science of inference; and the other, whether it furnishes a definite and certain method for the solution of these.

The former of these questions may, with perfect confidence, be answered in the affirmative. It admits of absolute demonstration, that there is no chain of valid inference which the ordinary logic is incompetent to express, or, in other words, which is not reducible to conversion or syllogism. Some logicians have been of opinion that conversion is nothing else than syllogism at bottom; but, for what we have at present in view, it is unnecessary to discuss this question. Suffice it to say, that, whether conversion and syllogism be substantially identical or not, all immediate inference is of the nature of conversion, and all mediate inference (or reasoning proper) of the nature of syllogism. Does Professor Boole deny this? Formally, and in plain terms. "Possibly," he writes, "it may here be said that the logic of Aristotle, in its rules of syllogism and conversion, sets forth the elementary processes of which all reasoning consists, and that beyond these there is neither scope nor occasion for a general method. I have no desire to point out the defects of the common logic, nor do I wish to refer to it any further than is necessary, in order to place in its true light the nature of the present treatise. With this end alone in view, I would remark: 1st. That syllogism, conversion, &c., are not the ultimate processes of logic. It will be shown in this treatise that they are founded upon, and are resolvable into, ulterior and more simple processes which constitute the real elements of method in logic. Nor is it true that all inference is reducible to the particular forms of syllogism and conversion. 2nd. If all inference were reducible to these processes alone (and it has been maintained that it is reducible to syllogism alone), there would still exist, &c." In illustration of the statement, that some inference is not reducible to the forms of syllogism and conversion, Professor Boole examines the case of conversion, and arrives at the result that "conversion is a particular application of a much more general process in logic, of which," he adds, "many examples have been given in this work." In like manner he examines the case of syllogism; and his conclusion is as follows: "Here, then, we have the means of definitely resolving the question, whether syllogism is indeed the fundamental type of reasoning,—whether the study of its laws is co-extensive with the study of deductive logic. For if it be so, some indication of the fact must be given in the system of equations upon the analysis of which we have been engaged. No sign, however, appears that the discussion of all systems of equations expressing propositions is involved in that of the particular system examined in this chapter. And yet writers on logic have been all but unanimous in their assertion, not merely of the supremacy, but of the universal sufficiency of syllogistic inference in deductive reasoning." These statements, that conversion and syllogism are branches of a much more general process, have of course no meaning except on the supposition that the "much more general process" is not reducible to conversion and syllogism. If reducible to these, it would not be a more general process. Now we take our stand firmly on the position, that a chain of valid reasoning, which cannot be broken into parts, every one of which shall be an instance either of conversion or of syllogism, is not possible. We are prepared to show this in the case of every one of the examples of his "more general process" which Professor Boole gives in his work. Nay, we go farther, and as was intimated above, hold it to be absolutely demonstrable, that, from the nature of the case, inference cannot be of any other description than conversion or syllogism.

To make this out, let it be remarked that the conclusion of an argument exhibits a relation between two terms, say X and Y. It is an important assumption in Professor Boole's doctrine, that a proposition may exhibit a relation between many terms. This is not exactly true. A proposition may involve a relation between a variety of terms implicitly; but explicitly exhibits a relation only between two. Take, for instance, the proposition—"Men who do not possess courage and practise self-denial are not heroes." Here, on Professor Boole's method, a variety of concepts are supposed to be before the mind, as, *men*, *those who practise self-denial*, *those who possess courage*, and *heroes*. But in reality, when we form the judgment expressed in the proposition given, the separate concepts, *men*, *those who practise self-denial*, *those who possess courage*, are not before the mind; but simply the two concepts, *men who do not possess courage and practise self-denial*, and *heroes*. What is a judgment but an act of comparison? And the comparison is essentially a comparison of two concepts, each of which may no doubt involve in its expression a plurality of concepts, but these necessarily bound together by the comparing mind into a unity. Now, if the conclusion of an argument exhibits a relation between two terms X and Y, this conclusion must be drawn (what other way is possible?) either through an immediate comparison of X and Y with one another, or by a mediate comparison of them through something else. If it be drawn by an immediate comparison of *X* and *Y*, then no concepts enter into the argument except *X* and *Y*, and the argument is reduced to conversion. But if the conclusion be drawn mediately, it must be by the comparison of *X* and *Y* with some third thing: not with a plurality of other things, but with some single thing. Here we have the mind drawing its inference in a syllogism. What the various admissible forms of conversion and syllogism may be, or whether these forms have been correctly specified by particular eminent logicians, are minor questions. The essential thing in a philosophical respect is, that the mind, in the inferences which it draws, does and can work in no other moulds than those described. All this seems to us so plain that we confess ourselves utterly puzzled to comprehend how men of profound and original genius have been beguiled into an assertion of the contrary.

Professor Boole himself, in summing up his assault on the Aristotelian Logic, comes very near admitting what we contend for. "As Syllogism," he says, "is a species of elimination, the question before us manifestly resolves itself into the two following ones: 1st. Whether all elimination is reducible to Syllogism; 2nd. Whether deductive reasoning can, with propriety, be regarded as consisting only of elimination. I believe, upon careful examination, the true answer to the former question to be, that it is always theoretically possible so to resolve and combine propositions that elimination may subsequently be effected by the syllogistic canons, but that the process of reduction would in many instances be constrained and unnatural, and would involve operations which are not syllogistic. To the second question I reply, that reasoning cannot, except by an arbitrary restriction of its meaning, be confined to the process of elimination." With regard to this second question, we merely note in passing, that we have proved in the preceding paragraph that inference, where not immediate or of the nature of conversion, can be nothing else than elimination. It is, however, with the first question, whether elimination is reducible to syllogism, that we have now more particularly to do; and we accept with satisfaction the admission, guarded and (to some extent) neutralised as it is, that every line of argument may be thrown into a form in which the eliminations that take place are effected by the syllogistic canons. It is quite irrelevant to notice, as Professor Boole does, that the process of reduction would, in many instances, be constrained and unnatural; for we are not here in the province of Rhetoric. Much more to the purpose is the charge, that the process of reduction would involve operations which are not syllogistic. The operations referred to are those embraced in the "much more general process" in which, as we have seen, our Author holds conversion and syllogism to be contained. Of course, the ground which we take in reply is, on the one hand, to challenge the production of an instance of valid inference, which cannot be reduced to either conversion or syllogism; and on the other hand, to fall back upon the demonstration which we have given of the absolute impossibility of valid inference being anything else than conversion or syllogism.

In stating the charge of incompleteness brought by our Author against the Aristotelian system, we explained his meaning to be, that, from the very nature of the system, there is an indefinite variety of problems belonging to the science of inference, which the system is incapable of solving, or for the solution of which, at all events, it furnishes no definite and certain method. We have, we trust, fully refuted the opinion that there are problems in the science of inference which the Aristotelian logic is incapable of solving. But Professor Boole urges, that, even if all inference were reducible to conversion and syllogism, "there would still exist the same necessity for a general method. For it would still be requisite to determine in what order the processes should succeed each other, as well as their particular nature, in order that the desired relation should be obtained. By the desired relation I mean that full relation which, in virtue of the premises, connects any elements selected out of the premises at will, and which, moreover, expresses that relation in any desired form and order. If we may judge from the mathematical sciences, which are the most perfect examples of method known, this directive function of method constitutes its chief office and distinction. The fundamental processes of arithmetic, for instance, are in themselves but the elements of a possible science. To assign their nature is the first business of its method, but to arrange their succession is its subsequent and higher function. In the more complex examples of logical deduction, and especially in those which form a basis for the solution of difficult questions in the theory of probabilities, the aid of a directive method, such as a Calculus alone can supply, is indispensable."

Now, we at once admit that the Aristotelian logic neither has, nor professes to have, any such method as that here described. But can it justly, on that account, be charged with incompleteness? A science must not, because it does not teach everything, be therefore reckoned incomplete: enough, if it teaches the whole of its own proper circle of truths. The special question which the scholastic logic proposes to itself is: what are the ultimate abstract forms according to which all the exercises of the discursive faculty proceed? The science is complete, because it furnishes a perfect answer to this question.

But, it may be said, is it not desirable to have a method enabling us certainly to determine, in every case, the relation which any of the concepts explicitly or implicitly entering into a group of premises bear to the others? Most desirable. And herein consists the real value of Professor Boole's labours. He has devised a brilliantly original Calculus by which he can, through processes as definite as those which the Algebraist applies to a system of equations, solve the most complicated problems in the science of inference—problems which, without the aid of some such Calculus, persons most thoroughly versed in the ordinary logic might have no idea how to treat. In expressing our dissent, as we have been obliged very strongly to do, from much that is contained in Professor Boole's treatise, we have no desire to rob that eminent writer of the credit justly belonging to him. Our wish has been simply to separate the chaff from the wheat, and to point out accurately what constitutes, as far as the "Investigation" is concerned, Professor Boole's claim to renown.

Our readers will, however, be now anxious to obtain some fuller information regarding the method about which so much has been said, and which is the same with "the more general process" under which the processes of the scholastic logic are held by Professor Boole to be comprehended. This part of our article must necessarily be altogether technical; and we shall require to ask our readers to take a few things on trust; but we hope to be able to present the subject in such a manner as to give at least some idea of the system we are to endeavour to describe. Those who desire to become thoroughly acquainted with it will of course study the "Investigation" for themselves.

We begin by referring to the development of logical functions. An expression which in any manner involves the concept x, is called a function of the concept, and is written $f(x)$. Now there is one standard form to which functions of every kind may be reduced. This form is not an arbitrary one, but is determined by the circumstance that every conceivable object must rank under one or other of the two contradictory classes x and $1 - x$. Hence every conceivable object is included in the expression,

$$ux + v(1 - x); \quad (8)$$

proper values being given to u and v. For, if a given concept belong to the class x, then, by making $v = 0$, the expression (8) becomes ux, which, by (1), means *some* x; and if the given concept belong to the class $1 - x$, then, by making $u = 0$, the expression (8) becomes $v(1 - x)$, which, by (1) and (6), means *some not* x. Therefore, $f(x)$ being any concept depending on x, we may put

$$f(x) = ux + v(1 - x) \quad (9)$$

It has been shown that one of the coefficients, u, v, must always be zero; but the forms of these coefficients may be determined more definitely. For, by making $x = 0$ in (9), the result is $v = f(0)$; and by making $x = 1$, there results $u = f(1)$; by substituting which values of u and v in (9), we get

$$f(x) = f(1)\,x + f(0)\,(1 - x) \quad (10)$$

This is the expansion or development of the function x. The expressions x, $1 - x$, are called the constituents of the expansion; and $f(1)$ and $f(0)$ are termed the coefficients. The same phraseology is employed when a function of two or more symbols is developed.

Any one in the least degree acquainted with mathematical processes will understand how the development of functions of two or more symbols can be derived from equation (10). In fact, by (10), we have

$$f(x, y) = f(1, y)\,x + f(0, y)\,(1 - x).$$

But again, by (10),

$$f(1, y) = f(1, 1)\,y + f(1, 0)\,(1 - y),$$

and

$$f(0, y) = f(0, 1)\,y + f(0, 0)\,(1 - y).$$

$$\therefore f(x, y) = f(1, 1)\,xy + f(1, 0)\,x(1 - y)$$
$$+ f(0, 1)\,y(1 - x) + f(0, 0)\,(1 - x)(1 - y) \quad (11)$$

The development of a function of three symbols may be written down, as we shall have occasion in the sequel to refer to it:

$$\begin{aligned} f(x, y, z) = f(1, 1, 1)\, xyz &+ f(1, 1, 0)\, xy\,(1-z) \\ &+ f(1, 0, 1)\, xz\,(1-y) + f(1, 0, 0)\, x\,(1-y)\,(1-z) \\ &+ f(0, 1, 1)\, yz\,(1-x) + f(0, 1, 0)\, y\,(1-x)\,(1-z) \\ &+ f(0, 0, 1)\, z\,(1-x)\,(1-y) \\ &+ f(0, 0, 0)\,(1-x)\,(1-y)\,(1-z) \quad \ldots\ldots\ldots(12) \end{aligned}$$

As the object of the expansion of logical symbols may not be evident at first sight, and as the process may consequently be regarded by some as barbarous, we may observe that not only is there a definite aim in the development, but the thing aimed at, has, in our opinion, been most felicitously accomplished. Of this our readers will probably be satisfied when they are introduced to some specimens of the use which is made of the formulæ obtained; in the meantime it may throw some light on the character of these formulæ if we notice that the constituents of an expansion represent the several exclusive divisions of what our author terms the universe of discourse, formed by the predication and denial in every possible way of the qualities denoted by the literal symbols. In the simplest case, that in which the function is one of a single concept, it will be seen by a glance at (10) that there are only two such possible ways, x and $1 - x$. In the case of a function of two symbols, there are [see (11)] four such ways, xy, $x(1-y)$, $y(1-x)$, $(1-x)(1-y)$. In a function of three symbols there are eight such ways; and so on. A development in which the constituents are of this kind prepares the way for ascertaining all the possible conclusions, in the way either of affirmation or denial, that can be deduced, regarding any concept, from any given relations between it and the other concepts.

If S be the sum of the constituents of an expansion, and P the product of any two of them, then

$$S = 1, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (13)$$

$$\text{and } P = 0. \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (14)$$

The truth of these beautiful and important propositions will easily be gathered by an intelligent reader from an inspection of the formulae, (10), (11), (12). Another important proposition is involved in (14), namely, that, if $f(x) = 0$, either the constituent or the coefficient in every term of the expansion of $f(x)$ must be zero. For, let

$$f(x) = Q + A\,X + A_1\,X_1 + \ldots\ldots\ldots + A_n\,X_n\,;$$

where A, A_1, &c., are the coefficients which are not zero, their corresponding constituents being X, X_1, *&c.*; while Q represents the sum of those terms in which the coefficients are zero. Then we say that

$$X = 0 \ldots\ldots\ldots\ldots (15)$$

For, since $Q = 0$, and $f(x)$ is supposed to vanish,

$$A X + A_1 X_1 + \&c. = 0$$

$$\therefore A X^2 + A_1 X X_1 + \&c. = 0$$

But, by (14), $X X_1 = X X_2 = \ldots\ldots = X X_n = 0$. Therefore

$$A X^2 = 0.$$

But A is not zero. Therefore X must be zero.

These principles having been laid down, our best course will probably now be to take a few examples, and to offer in connection with them such explanations as may seem necessary of the mode of procedure which they are intended to illustrate.

Our first example shall be one in which but a single proposition is given: "clean beasts are those which both divide the hoof and chew the cud." Let

x = clean beasts,
y = beasts dividing the hoof,
z = beasts chewing the cud.

Then, the given proposition, symbolically expressed, is,

$$x = y z,$$

or, by transposition,

$$x - y z = 0 \ldots\ldots\ldots\ldots (16).$$

This premiss contains a relation between three concepts; and, according to Professor Boole, a properly constructed science of inference should enable us, by some defined process, to show what consequence, as respects any one of these, follows from the premiss. Now, the definite and invariable process which Professor Boole applies, with the design which has been indicated, to an equation such as (16), is to develop the first member of the equation. Writing, then,

$$f(x, y, z) = x - y z,$$

we have, $f(1, 1, 1) = 0$,

$$f(0, 0, 0) = 0,$$

and so on. Hence [see (12)] the developement required is

$$\begin{aligned} x - y z = {} & x y (1 - z) + x z (1 - y) \\ & + x (1 - y)(1 - z) - y z (1 - x) \\ & + 0 x y z + 0 y (1 - x)(1 - z) \\ & + 0 z (1 - x)(1 - y) \\ & + 0 (1 - x)(1 - y)(1 - z). \end{aligned}$$

Therefore, by (16),

$$x y (1-z) + x z (1-y) + x (1-y) (1-z) - y z (1-x) = 0:$$

and therefore, by (15),

$$\left.\begin{aligned} x y (1 - z) &= 0. \\ x z (1 - y) &= 0, \\ x (1 - y) (1 - z) &= 0, \\ y z (1 - x) &= 0. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (17)$$

Still farther, since, by (13), the sum of the constitutents of an expansion is unity; and since four of the constituents in the expansion of $x - y z$ have been shewn to be zero; it follows that the sum of the remaining constituents in the expansion of $x - y z$ is unity. That is,

$$\begin{aligned} x y z + y (1 - x) (1 - z) + z (1 - x) (1 - y) \\ + (1 - x) (1 - y) (1 - z) = 1. \ \ldots\ldots (18) \end{aligned}$$

It is obvious that this method can be applied in every case. To what then does it lead? First of all, in the group of equations (17), we have brought before us all the different classes (if the expression may be permitted) to which the given proposition warrants us in saying that nothing can belong; and next, in equation (18) we have brought before us those different classes to one or other of which the given proposition warrants us in asserting that everything must belong. For instance, the first of equations (17) denies the existence of beasts which are clean (x) and divide the hoof (y) but do not chew the cud ($1 - z$); the second denies the existence of beasts which are clean (x) and chew the cud (z) but do not divide the hoof ($1 - y$); and so on. Equation (18), again, informs us that the universe, which is represented by 1, is made up of four classes, in one or other of which therefore every thing must rank; the first denoted by $x y z$, the second by $y (1 - x) (1 - z)$; and so on. As an example of the interpretation of the expressions by which these classes are denoted, we may take the last, $(1 - x) (1 - y) (1 - z)$. This represents things which are neither clean beasts, nor beasts chewing the cud, nor beasts dividing the hoof.

By the method employed, we have been able to indicate certain classes which do not exist, and also to indicate certain classes in one or other of which every thing existing is found. But this, it may be said, is not a solution of the most general problem of inference. The most general problem is: to express (speaking mathematically) any one of the symbols entering into the given premiss, or any function thereof, as an explicit function of the others. To the problem as put even thus in its widest generality, Professor Boole's processes extend. It would make our article too lengthened were we to go into minute details; but we must endeavour to give some idea of the course here followed, as it both is extremely interesting as a matter of pure speculation, and forms an important part of the system under consideration.

Take the equation in (16), $x - yz = 0$; and, as a simple instance will serve the purpose of illustration as well as a complicated one, let the inquiry be: how can z be expressed in terms of x and y? In ordinary Algebra we should have

$$z = \frac{x}{y} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(19)$$

But though both sides of an equation may, in Logic as in Algebra, be multiplied (so to speak) by the same quantity, they cannot, in Logic, be legitimately divided by the same quantity. For instance, let the objects common to the class X and to the class U be identical with those common to the class Y and to the class U; in other words, let

$$UX = UY;$$

it does not follow that X is identical with Y, or symbolically, that

$$X = Y.$$

Hence equation (19) could not, in Logic, be legitimately deduced from (16), even if y were an explicit factor of x. But still further, when x has not y as one of its factors, the expression $\frac{x}{y}$ is not, in the logical system, interpretable. Nevertheless, Professor Boole shows that conclusions both interpretable and correct will *ultimately* be arrived at, if the value of z be deduced Algebraically, as in (19), and the expression $\frac{x}{y}$ be then, as a logical function, subjected to development. Now, if $\frac{x}{y}$ be developed by (11), and the expansion equated to z, we get

$$z = xy + \tfrac{1}{0}x(1-y) + 0(1-x)y + \tfrac{0}{0}(1-x)(1-y) \ldots\ldots(20)$$

Here we have two symbols, $\frac{0}{0}$ and $\frac{1}{0}$, the meaning of which has not yet been determined. Our author shows that the former, which in Algebra denotes an indefinite numerical quantity, denotes in the logical system an indefinite class. In Algebra $\frac{1}{0}$ denotes infinity; and, as is well known, when it occurs as the co-efficient in a term in an equation all of whose other terms are finite, this indicates that the quantity of which it is the co-efficient is zero. So, in the logical system, if, in any term of an equation obtained in the manner in which equation (20) has been obtained, the co-efficient be $\frac{1}{0}$, the corresponding constituent must be 0. These are certainly very remarkable analogies. But let us see what follows. We have first, from (20),

$$x(1-y) = 0.$$

Hence as the equation (20) describes the separate classes of which z consists, and as there is no such class as $x(1-y)$ in existence, the second term on the right hand side of equation (20) may be rejected. The third term also may be omitted, its co-efficient being zero. This reduces the equation to the form,

$$z = xy + \tfrac{0}{0}(1-x)(1-y):$$

which means, that beasts which chew the cud consist of the class xy, together with an indefinite remainder of beasts common to the classes $1-x$ and $1-y$.

Before leaving the subject of inference from a single premiss, we must say a few words regarding elimination; for though, in Algebra, elimination is possible only when two or more equations are given, Professor Boole, shows that, in Logic, a class symbol may be eliminated from a single equation. In fact, elimination from two or more premises is ultimately reduced by our author to elimination from a single premiss. And yet, as if to preserve the analogy between Algebra and Logic, even where the two sciences seem to differ most widely from one another, the possibility of eliminating x from a single premiss in the latter science, arises from the circumstance, that, in that science the equation previously referred to as expressing the Law of Duality always subsists; and it is by the combination of that equation with the given proposition that the elimination of x from the given proposition is effected. For let the given proposition be

$$f(x) = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots (21)$$

Then, by (10),

$$f(1)x + f(0)(1-x) = 0.$$

$$\therefore x\{f(0) - f(1)\} = f(0),$$

$$\text{and, } (1-x)\{f(0) - f(1)\} = -f(1).$$

$$\therefore x(1-x)\{f(0) - f(1)\}^2 = -f(0)f(1).$$

But, by the Law of Duality, $x(1-x) = 0$. Therefore

$$f(0)f(1) = 0 \text{ .... (22)}$$

which is the result of the elimination of x from equation (21). We cannot pause to give examples of the use of the formula (22); but we must quote an interpretation of it, viewed as the result of the elimination of x from (21), which strikes us as extremely elegant. The formula implies that either $f(0) = 0$, or $f(1) = 0$. Now the latter equation $f(1) = 0$ expresses what the given proposition $f(x) = 0$ would become if x made up the universe; and the former $f(0) = 0$ expresses what the given proposition would become if x had no existence. Hence, (22) being derived from (21), it follows that *what is equally true whether a given class of objects embraces the whole universe or disappears from existence, is independent of that class altogether.*

The principle of elimination is extended by our author to groups of equations, by the following process. Let

$$\left.\begin{array}{l} T = 0, \\ V = 0, \\ U = 0, \\ \ldots\ldots\ldots \end{array}\right\} \text{ .... (23)}$$

be a series of equations, in which T, U, V, &c., are functions of the concept x. Then

$$T^2 + V^2 + U^2 + \&c. = 0 \text{ .... (24)}$$

It is shown by Professor Boole that the combined interpretation of the system of equations (23) is involved in the single equation (24). Indeed, had all the terms in the developments of T, V, U, &c., been such as to satisfy the Law of Duality, it would have been sufficient to have written

$$T + V + U + \&c. = 0.$$

In order now to eliminate x from the group (23), it is sufficient to eliminate it, by the method described in the preceding paragraph, from the single equation (24); and, if the result be

$$W = 0,$$

this equation will involve all the conclusions that can legitimately be derived from the series of equations (23) with regard to the mutual relations of the concepts, exclusive of x, which enter into these equations.

We do not see how it is possible for any one not blinded by prejudice against every thing like an alliance of Logic with formulæ and processes of a mathematical aspect to deny that these are very remarkable principles. By way of instance, we select from the work under review the following problem, in which two premises are given. Let it be granted, first, that the annelida are soft-bodied, and either naked or enclosed in a tube; and, next, that they consist of all invertebrate animals having red blood in a double system of circulating vessels. Put

A = annelida, s = soft-bodied animals,
n = naked, t = enclosed in a tube,
i = invertebrate, r = having red blood in &c.

Then the given premises are

$$A = vs\left\{n(1-t) + t(1-n)\right\}, \ldots\ldots(25)$$
$$A = ir \ldots\ldots(26)$$

Suppose the problem then to be: to find the relation in which soft bodied animals enclosed in tubes stand to the following elements, viz., the possession of red blood, of an external covering, and of a vertebral column. Professor Boole would doubtless have granted that this problem admits of being solved by what he calls the ordinary logic; but he would probably have contended that the ordinary logic does not possess any definite and invariable method of solution. A skilful thinker may be able to find out how syllogisms may be formed so as ultimately to give him the relation which soft bodied animals enclosed in tubes bear to the elements specified; but what of thinkers who are not very skilful? How are they to proceed? In Professor Boole's system, the process is as determinate, and as certain of leading to the desired result, as the rules for solving a group of simple equations in Algebra. Eliminate v, the symbol of indefinite quantity, from (25). Reduce (25), thus modified, and (26), to a single equation, by the method described in a previous paragraph. The equation is

$$A\left\{1 - sn(1-t) - st(1-n)\right\} + A(1-ir) + ir(1-A) + nt = 0.$$

Then, since the annelida are not to appear in the conclusion, we must eliminate A, by (22), from this equation. This will be found to give us

$$ir\left\{1 - sn(1-t) - st(1-n)\right\} + nt = 0.$$

And ultimately we get

$$st = ir(1-n) + \tfrac{0}{0}\,i(1-r)(1-n) + \tfrac{0}{0}(1-i)(1-n);$$

the interpretation of which is: *Soft bodied animals enclosed in tubes* *(st) consist of all invertebrate animals having red blood (ir) and not naked* $(1 - n)$, *and an indefinite remainder* $(\frac{0}{0})$ *of invertebrate animals (i) not having red blood* $(i - r)$ *and not naked* $(1 - n)$ *and of vertebrate animals* $(1 - i)$ *which are not naked* $(1 - n)$.

We have entered so fully into the explanation of Professor Boole's system in its bearing on what he terms Primary (virtually equivalent to Categorical) Propositions, that we cannot follow him into the field of Secondary (virtually equivalent to Conditional, that is, Disjunctive and Hypothetical) Propositions. Nor is it necessary that we should do so; for our object is not to give a synopsis of the "Investigation," but simply to make the nature of the work understood; and, for that purpose, what has been said is sufficient. The application of the Calculus to Secondary Propositions is exceedingly similar, in respect not only of the general method followed, but even of the particular formulæ obtained, to its application to Primary. All that is peculiar in the treatment of Secondary Propositions arises from the introduction of the idea of Time. For instance, the proposition, "If X is Y, A is B," is held to be not substantially different in meaning from this: "the time in which X is Y, is time in which A is B." Such being the fundamental view taken, symbols like x and y are used to represent the portions of time in which certain propositions (e.g., X is Y, A is B) are true. Then, the symbol 1 denoting the universe of Time, or Eternity, the expressions, $1 - x$, $1 - y$, will denote those portions of time respectively in which the propositions, X is Y, A is B, are not true; and so on.

The extension of his method, by Professor Boole, to the theory of Probabilities, is a splendid effort of genius on the part of the author, and furnishes a most convincing illustration of the capabilities of the method. The part of the "Investigation" which is devoted to this subject, is much too abstruse to admit of being here more particularly considered; but, to show what the method can accomplish —though the bow of Ulysses perhaps needs the arm of Ulysses to bend it—we may simply state one of the problems of which Professor Boole gives the solution. "If an event can only happen as a consequence of one or more of certain causes, $A_1, A_2, \ldots\ldots, A_n$, and if generally C_1 represents the probability of the cause A_1, and p_1 the probability that, if the cause A_1 exist, the event E will occur, then the series of C_1 and p_1 being given, required the probability of the event E."

To those who have followed us thus far, it will be evident what final judgment we are to pass on the work under review. On the one hand, as a contribution to philosophy, in the strict sense of that term, it does not possess any value. Professor Boole distinctly, though modestly enough, avows the opinion, that, in his "Investigation," he has gone deeper than any previous inquirers into the principles of discursive thinking, and that he has thus thrown new light on the constitution of the human mind. We are sorry to be unable to accept this view. But, on the other hand, Professor Boole is entitled to the praise of having devised a Method, according to which, through definite processes, it can be ascertained what conclusions, regarding any of the concepts entering into a system of premises, admit of being drawn from these premises. This Method depends on a Calculus, original, ingenious, singularly beautiful both in itself and in its relations to the science of Algebra, and capable (in hands like those of its inventor) of striking and important applications. In a word, the merit of the Treatise lies in that part of it which has nothing to do with the Laws of Thought, but which is devoted to showing how inferences, from data however numerous and complicated, and whatever be the matter of the discourse, can be reached through definite mathematical processes.